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On the stability of a rigid body in a magnetostatic equilibrium

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Abstract

We study the stability of a perfectly conducting body in a magnetostatic equilibrium. The body is immersed in a fluid which is threaded by a three-dimensional magnetic field. The fluid may be perfectly conducting, non-conducting or have finite conductivity. We generalise the classical stability criterion of Bernstein et al. (Proc. Roy. Soc. London Ser. A 244 (1958) 17–40; I.B. Bernstein, The variational principle for problems of ideal magnetohydrodynamic stability, in: A.A. Galeev, R.N. Sudan (Eds.), Basic Plasma Physics: Selected Chapters, North-Holland, Amsterdam, 1989, pp. 199–227) and show that the body is stable to small isomagnetic perturbations if and only if the magnetic energy has a minimum at the equilibrium. For an equilibrium of a body in potential magnetic field, we obtain a sufficient condition for genuine nonlinear stability.

1. Introduction

Let us start with a simple problem. In two dimensions, we consider a perfectly conducting ellipse which sits in an incompressible, inviscid, perfectly conducting fluid. Suppose that there exists an irrotational magnetic field throughout the fluid which is uniform at infinity and whose field lines are tangent to the boundary of the ellipse. Its centre is fixed but the ellipse is free to rotate. It is evident from symmetry that there are two equilibrium positions for the the ellipse, corresponding to the longer axis of the ellipse being either parallel or perpendicular to the magnetic field at infinity $\mathbf{H}_0$ (Fig. 1). A natural question to ask is: 'are these equilibria stable?'. Below we present simple physical arguments that give an answer to this question.

Consider first the case where the longer axis is parallel to $\mathbf{H}_0$ (Fig. 1(a)). If the ellipse is rotated slightly, then the field lines around it become stretched and the magnetic energy rises. This suggests that the magnetic energy has a local minimum in the equilibrium and, as a consequence, the equilibrium is stable.

When the longer axis is perpendicular to $\mathbf{H}_0$ (Fig. 1(b)), a similar argument suggests that the ellipse is unstable. That is, a slight rotation of the ellipse allows the magnetic field to relax (i.e., the field lines contract), so that the magnetic energy decreases. This indicates that the magnetic energy has a local maximum in the equilibrium and this may lead to instability.

This simple problem illustrates how energy arguments provide a convenient framework for analysing the stability of a body in a magnetostatic equilibrium. The ellipse is stable if the magnetic energy has a mininmum at equilibrium and unstable if it has a maximum. Note that the first conclusion (about stability) is always true, while the second one (about instability), may sometimes be erroneous. In general, instability of an equilibrium of a mechanical system does not simply follow from the absence of a local minimum of the energy at this equilibrium. Even though conclusions about instability in such cases seem physically reasonable and are valid for many particular situations, we cannot assert *a priori* that it is true in the general situation. A classical example from finite-dimensional mechanics when the system has no minimum at equilibrium but this equilibrium is

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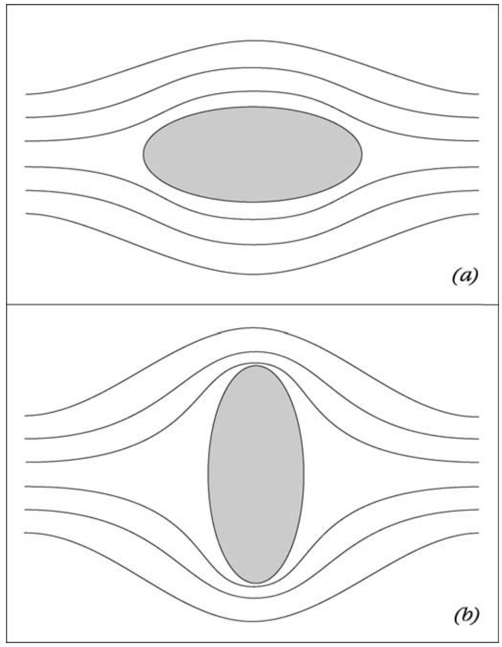

Fig. 1.

stable may be found in [1]. In continuum mechanics the situation is much more complicated and it is always necessary to prove the corresponding proposition.

In this paper we generalise the above arguments about the stability and instability of the ellipse and provide necessary and sufficient conditions for linear stability of an arbitrary three-dimensional body immersed in a fluid. Initially we restrict ourselves to an inviscid, perfectly conducting fluid. Then we generalise our theory to a viscous fluid and to a fluid with finite electric conductivity. The case of a non-conducting fluid is also considered. The dissipation effects (due to viscosity and conductivity) as well as the case of a non-conducting fluid are included in order to understand whether the stability results for a perfectly conducting body in a perfectly conducting inviscid fluid can be of certain value for real fluids. The domain of applicability of the results is discussed in Section 4 of the paper.

Our main results concern the linear stability. However, in the case of a body in potential magnetic field, we give a sufficient condition for genuine nonlinear stability. We consider this result as one of the most valuable outcomes of the paper, because there are virtually no nonlinear stability criteria in problems with moving boundaries.

Our findings may be viewed as a generalisation of the linear stability criterion of Bernstein et al. [2] and a development of the ideas and results contained in the papers by Barston [3], Moffatt [4], Vladimirov and Rumyantsev [5], Davidson [6] and Vladimirov and Ilin [7]. The novelty of the work lies in the inclusion of: (i) a rigid body within the fluid; (ii) viscosity; (iii) finite conductivity; (iv) finite-amplitude disturbances.

2. Inviscid, perfectly conducting fluid

2.1. Formulation of the problem

Consider a perfectly conducting, incompressible, inviscid fluid, which occupies a closed domain $\mathcal{D}$. Within the fluid sits a perfectly conducting, rigid body, $\mathcal{D}_b$. The fluid is pervaded by a magnetic field whose field lines are tangent to the outer surface and the surface of the body. The entire configuration is in equilibrium (both the fluid and the body are at rest) and the problem is to determine the stability of the equilibrium.

Let us introduce some notation. We use $\mathcal{D}_f$ to denote the domain occupied by the fluid ($\mathcal{D}_f = \mathcal{D} - \mathcal{D}_b$). Its boundary $\partial\mathcal{D}_f$ consist of two parts: the inner boundary $\partial\mathcal{D}_b$ representing the surface of the rigid body and the outer boundary $\partial\mathcal{D}$ which is assumed to be fixed in the space and perfectly conducting. Throughout the paper, we use Cartesian coordinates with the origin at the center of mass of the body in its equilibrium position. The unit normal $\mathbf{n}$ points outward from $\mathcal{D}$ and $\mathcal{D}_b$. For simplicity, we take the density of the fluid to be equal 1 and measure the magnetic field $\mathbf{H}$ in Alfven velocity units. We assume that there are no external forces applied to the rigid body and to the fluid. (It is not difficult to include them into consideration if necessary.)

In the basic equilibrium, the magnetic field $\mathbf{H}(\mathbf{x})$ satisfies the equations

$$\mathbf{J}\times\mathbf{H}=\nabla P,\quad \mathbf{J}=\nabla\times\mathbf{H},\quad \nabla\cdot\mathbf{H}=0\quad \text{in } \mathcal{D}_f \tag{2.1}$$

(where $\mathbf{J}$ is the electric current density and P is the pressure) and boundary conditions

$$\mathbf{H}\cdot\mathbf{n}=0\quad \text{at } \partial\mathcal{D} \text{ and } \partial\mathcal{D}_b. \tag{2.2}$$

The total force and torque exerted on the body by the fluid and the magnetic field are zero in the equilibrium, so that

$$\mathbf{F}=\int\limits_{\partial\mathcal{D}_b}\left(P+\frac{\mathbf{H}^2}{2}\right)\mathbf{n}\,\mathrm{d}S=0,\qquad \mathbf{N}=\int\limits_{\partial\mathcal{D}_b}\left(P+\frac{\mathbf{H}^2}{2}\right)\mathbf{x}\times\mathbf{n}\,\mathrm{d}S=0. \tag{2.3}$$

(Here $\mathbf{x}$ represents a point on the body surface $\partial\mathcal{D}_b$.)

We are interested in the behaviour of small perturbations involving movements of both the fluid and the body. The equations governing the evolution of small perturbation of the velocity $\mathbf{u}(\mathbf{x},t)$ and the magnetic field $\mathbf{h}(\mathbf{x},t)$ are the standard equations of ideal magnetohydrodynamics linearised in a neighbourhood of the basic equilibrium (2.1), (2.2):

$$\mathbf{u}_t=-\nabla p+\mathbf{J}\times\mathbf{h}+\mathbf{j}\times\mathbf{H}, \tag{2.4}$$

$$\mathbf{h}_t=[\mathbf{u},\mathbf{H}]\equiv(\mathbf{H}\cdot\nabla)\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{H}, \tag{2.5}$$

$$\nabla\cdot\mathbf{u}=\nabla\cdot\mathbf{h}=0. \tag{2.6}$$

In Eqs. (2.4)–(2.6), p is the perturbation pressure.

The nonlinear equations of motion of a rigid body in an inviscid non-conducting fluid and the linearization procedure are discussed in detail in [7]. Inclusion of the magnetic field is equivalent to adding the magnetic pressure to the purely hydrodynamic pressure in the expression for the force and torque acting on the body. Therefore, to save space, we do not discuss the nonlinear equations of motion here and refer the reader to the paper [7].

Small deviations of the rigid body from its equilibrium position are described by the displacement of its center of mass $\mathbf{r}$ and infinitesimal rotation characterized by the vector $\boldsymbol{\phi}=(\phi_1,\phi_2,\phi_3)$,[1] so that the displacement of the points of the body $\boldsymbol{\xi}_b$ is given by

$$\boldsymbol{\xi}_b=\mathbf{r}+\boldsymbol{\phi}\times\mathbf{x}. \tag{2.7}$$

The equations governing the evolution of $\mathbf{r}$ and $\boldsymbol{\phi}$ are the standard equations of classical mechanics linearised in a neighbourhood of the equilibrium:

$$M\ddot{\mathbf{r}}=-\int\limits_{\partial\mathcal{D}_b}\left\{p+\mathbf{H}\cdot\mathbf{h}+\boldsymbol{\xi}_b\cdot\nabla\left(P+\frac{\mathbf{H}^2}{2}\right)\right\}\mathbf{n}\,\mathrm{d}S, \tag{2.8}$$

$$I_{ik}\ddot{\phi}_k=-\int\limits_{\partial\mathcal{D}_b}\left\{p+\mathbf{H}\cdot\mathbf{h}+\boldsymbol{\xi}_b\cdot\nabla\left(P+\frac{\mathbf{H}^2}{2}\right)\right\}(\mathbf{x}\times\mathbf{n})_i\,\mathrm{d}S. \tag{2.9}$$

Here M is the mass of the body, I_{ik} is the moment of inertia tensor. Throughout the paper $\mathcal{D}_b$ and $\partial\mathcal{D}_b$ correspond to the equilibrium position of the body. Boundary conditions for the perturbation velocity $\mathbf{u}$ at $\partial\mathcal{D}$ and $\partial\mathcal{D}_b$ are the standard conditions of no normal flow through the rigid boundaries:

$$\mathbf{u}\cdot\mathbf{n}=0\quad\text{at } \partial\mathcal{D},\qquad \mathbf{u}\cdot\mathbf{n}=\mathbf{v}_b\cdot\mathbf{n}\quad\text{at } \partial\mathcal{D}_b, \tag{2.10}$$

where

$$\mathbf{v}_b=\dot{\mathbf{r}}+\dot{\boldsymbol{\phi}}\times\mathbf{x}$$

is the velocity of the points of the body.

[1] For example, the rotation of the body may be viewed as the result of an initial rotation about the original z axis through an angle ϕ_3, a second rotation about the intermediate y axis through an angle ϕ_2, and a third rotation about the final x axis through an angle ϕ_1.

We assume that the magnetic field is always tangent to the (perfectly conducting) boundaries $\partial\mathcal{D}$ and $\partial\mathcal{D}_b$, so that the boundary conditions for the perturbation magnetic field are

$$\mathbf{h}\cdot\mathbf{n}=0 \quad \text{at } \partial\mathcal{D}, \qquad \mathbf{h}\cdot\mathbf{n}=[\boldsymbol{\xi}_b,\mathbf{H}]\cdot\mathbf{n} \quad \text{at } \partial\mathcal{D}_b. \tag{2.11}$$

The boundary condition for the perturbation magnetic field at the body surface is obtained by linearisation of the exact condition of no normal magnetic field at the moving perfecttly conducting surface $\partial\widetilde{\mathcal{D}}_b(t)$. Its derivation is similar to the derivation of the linearised boundary condition for velocity in the case of a rigid body in a steady inviscid incompressible flow (see, e.g., [7]).

Following Bernstein et al. [2], we introduce Lagrangian displacement of fluid particles defined by the equations

$$\begin{aligned} &\mathbf{u}=\boldsymbol{\xi}_t, \quad \nabla\cdot\boldsymbol{\xi}=0 \quad \text{in } \mathcal{D},\\ &\boldsymbol{\xi}\cdot\mathbf{n}=0 \quad \text{at } \partial\mathcal{D}, \qquad \boldsymbol{\xi}\cdot\mathbf{n}=\boldsymbol{\xi}_b\cdot\mathbf{n} \quad \text{at } \partial\mathcal{D}_b \end{aligned} \tag{2.12}$$

and consider a particular class of perturbations – isomagnetic perturbations defined by the formula

$$\mathbf{h}=[\boldsymbol{\xi},\mathbf{H}]. \tag{2.13}$$

Eq. (2.13) represents the restriction on the initial data for the perturbation magnetic field. If it is satisfied at $t=0$, then it holds for all $t>0$. In what follows, we consider only isomagnetic perturbations.

Note that, in general, even if an equilibrium is stable to isomagnetic parturbations, it may be unstable to non-isomagnetic perturbations. It can be shown, however, that possible instability to non-isomagnetic perturbations can only be a slow algebraic instability, namely, the perturbation velocity cannot grow faster than linearly with time. Therefore, our restriction on the class of perturbations is equivalent to filtering out slow algebraic instabilities and our stability criteria should be understood as the results on existense or nonexistence of exponentially growing modes. The same approach had been adopted by Bernstein et al. [2].

With help of (2.13), Eq. (2.4) can be written as

$$\boldsymbol{\xi}_{tt}=-\nabla p+\mathbf{J}\times[\boldsymbol{\xi},\mathbf{H}]+\big(\nabla\times[\boldsymbol{\xi},\mathbf{H}]\big)\times\mathbf{H}. \tag{2.14}$$

Note that boundary conditions (2.11) for the perturbation magnetic field are automatically satisfied provided that we have Eq. (2.13) and boundary conditions for $\boldsymbol{\xi}$ (2.12).

Eqs. (2.14), (2.8), (2.9) and boundary conditions (2.12) give us the complete system of linearised equations governing the evolution of small perturbations of the equilibrium (2.1)–(2.3).

2.2. *Generalisation of the energy principle of Bernstein et al.*

It is convenient to rewrite Eq. (2.14) in a different, though equivalent, form. It can be shown that

$$\mathbf{J}\times[\boldsymbol{\xi},\mathbf{H}]+\big(\nabla\times[\boldsymbol{\xi},\mathbf{H}]\big)\times\mathbf{H}=(\mathbf{H}\cdot\nabla)\big\{(\mathbf{H}\cdot\nabla)\boldsymbol{\xi}\big\}-\nabla\big(\mathbf{H}\cdot[\boldsymbol{\xi},\mathbf{H}]\big)-(\boldsymbol{\xi}\cdot\nabla)\nabla\widetilde{P},$$

where

$$\widetilde{P}=P+\mathbf{H}^2/2$$

is the modified pressure in the basic equilibrium. Therefore, Eq. (2.14) can be written as

$$\boldsymbol{\xi}_{tt}=-\nabla\big(p+\mathbf{H}\cdot[\boldsymbol{\xi},\mathbf{H}]\big)+(\mathbf{H}\cdot\nabla)\big\{(\mathbf{H}\cdot\nabla)\boldsymbol{\xi}\big\}-(\boldsymbol{\xi}\cdot\nabla)\nabla\widetilde{P}. \tag{2.15}$$

Taking the inner product of (2.15) with $\boldsymbol{\xi}_t$ and integrating over $\mathcal{D}_f$ yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{D}_f}\left(\frac{1}{2}\boldsymbol{\xi}_t^2+\frac{1}{2}\big\{(\mathbf{H}\cdot\nabla)\boldsymbol{\xi}\big\}^2+\frac{1}{2}\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla\widetilde{P}\right)\mathrm{d}V=\int\int\limits_{\partial\mathcal{D}_b}(\boldsymbol{\xi}_t\cdot\mathbf{n})\big(p+\mathbf{H}\cdot[\boldsymbol{\xi},\mathbf{H}]\big)\,\mathrm{d}S.$$

Similarly, from Eqs. (2.8) and (2.9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}M\dot{\mathbf{r}}^2+\frac{1}{2}I_{ik}\dot{\phi}_i\dot{\phi}_k\right)=-\int\limits_{\partial\mathcal{D}_b}\big(p+\mathbf{H}\cdot[\boldsymbol{\xi},\mathbf{H}]+\boldsymbol{\xi}_b\cdot\nabla\widetilde{P}\big)(\mathbf{v}_b\cdot\mathbf{n})\,\mathrm{d}S.$$

It follows that the linearized equations conserve the quantity

$$E\equiv\frac{1}{2}\int\limits_{\mathcal{D}_f}\big(\boldsymbol{\xi}_t^2+\big\{(\mathbf{H}\cdot\nabla)\boldsymbol{\xi}\big\}^2+\boldsymbol{\xi}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla\widetilde{P}\big)\,\mathrm{d}V+\frac{1}{2}\int\limits_{\partial\mathcal{D}_b}\big(\boldsymbol{\xi}_b\cdot\nabla\widetilde{P}\big)(\boldsymbol{\xi}_b\cdot\mathbf{n})\,\mathrm{d}S+\frac{1}{2}M\dot{\mathbf{r}}^2+\frac{1}{2}I_{ik}\dot{\phi}_i\dot{\phi}_k \tag{2.16}$$

which may be interpreted as the energy of the linearized problem.

If E as a quadratic functional of $\boldsymbol{\xi}_t$, $\boldsymbol{\xi}$, $\mathbf{r}$, $\dot{\mathbf{r}}$, $\boldsymbol{\phi}$ and $\dot{\boldsymbol{\phi}}$ is positive definite, $\sqrt{E}$ can be taken as a norm to measure the deviation of perturbed flow from unperturbed one, and the conservation of E implies the stability of the basic state to small perturbations. In this case, the conservation of E implies the stability of the basic equilibrium: $E(t) = E(0)$.

Evidently, E would be positive definite if the 'potential energy'

$$W = \frac{1}{2}\int\limits_{\mathcal{D}_f} \left(\left\{(\mathbf{H}\cdot\nabla)\boldsymbol{\xi}\right\}^2 + \boldsymbol{\xi}\cdot(\boldsymbol{\xi}\cdot\nabla)\nabla\widetilde{P}\right)\mathrm{d}V + \frac{1}{2}\int\limits_{\partial\mathcal{D}_b}\left(\boldsymbol{\xi}_b\cdot\nabla\widetilde{P}\right)(\boldsymbol{\xi}_b\cdot\mathbf{n})\,\mathrm{d}S \tag{2.17}$$

is positive definite. The linear stability problem thus reduces to the analysis of W.

Note that W can be written in the equivalent form

$$W = \frac{1}{2}\int\limits_{\mathcal{D}_f}\left(\mathbf{h}^2 + \mathbf{h}\cdot(\mathbf{J}\times\boldsymbol{\xi})\right)\mathrm{d}V + \frac{1}{2}\int\limits_{\partial\mathcal{D}_b}\left(\boldsymbol{\xi}_b\cdot\nabla\widetilde{P}\right)(\boldsymbol{\xi}_b\cdot\mathbf{n})\,\mathrm{d}S, \tag{2.18}$$

where $\mathbf{h}$ is given by Eq. (2.13). In the case when the body is fixed, the surface integral in (2.17)/(2.18) vanishes, and we obtain the classic 'potential energy' of Bernstein et al. [2].

In the special case of a two-dimensional geometry and a planar magnetic field, the basic magnetic field $\mathbf{H}(x, y)$ and the perturbation magnetic field $\mathbf{h}(x, y)$ can be written in terms of the flux functions $A(x, y)$ and $a(x, y, t)$ as

$$\mathbf{H} = \nabla\times(A\mathbf{e}_z), \qquad \mathbf{h} = \nabla\times(a\mathbf{e}_z). \tag{2.19}$$

Eq. (2.13) reduces to

$$a = -\boldsymbol{\xi}\cdot\nabla A, \tag{2.20}$$

and the 'potential energy' (2.18) simplifies to

$$W = \frac{1}{2}\int\limits_{\mathcal{D}_f}\left((\nabla a)^2 + (\boldsymbol{\xi}\cdot\nabla A)(\boldsymbol{\xi}\cdot\nabla)\nabla^2 A\right)\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int\limits_{\partial\mathcal{D}_b}(\boldsymbol{\xi}_b\cdot\mathbf{n})(\boldsymbol{\xi}_b\cdot\nabla)\frac{\mathbf{H}^2}{2}\,\mathrm{d}l, \tag{2.21}$$

where $\boldsymbol{\xi}_b = \mathbf{r} + \phi\mathbf{e}_z\times\mathbf{x}$ is a two-dimensional vector.

It can be shown that W (both in three-dimensional and two-dimensional cases) is in fact the second variation of the magnetic energy on the set of all isomagnetic fields. The positive definiteness of W for a given equilibrium means therefore that the magnetic energy has a local minimum in the equilibrium.

So far we have obtained only a sufficient condition for stability. That is the equilibrium is stable provided that W is positive difinite. It can be shown that this condition is in fact both necessary and sufficient for stability, i.e., if W can be negative for some perturbation, then the equilibrium is unstable. We shall not give a proof of this fact here, since a similar proposition is proved in the next section in a more general setting (for a viscous, resistive fluid).

2.3. Examples

We now present two examples which illustrate our stability criterion.

(A) A circular cylinder surrounded by circular field lines. Consider the two-dimensional configuration shown in Fig. 2. Here we use polar coordinates (r, θ) and the magnetic field has only azimuthal component

$$\mathbf{H} = H_0(r)\mathbf{e}_\theta \quad \text{or, equivalently,} \quad A = A(r) \quad \text{with } A'(r) = H_0(r). \tag{2.22}$$

Let

$$\boldsymbol{\xi} = \frac{1}{r}\frac{\partial\chi}{\partial\theta}\mathbf{e}_r + \left(-\frac{\partial\chi}{\partial r}\right)\mathbf{e}_\theta$$

and

$$\chi(r,\theta) = \sum_{m=0}^{\infty}\left(\chi_m(r)\,\mathrm{e}^{\mathrm{i}m\theta} + \chi_m^*(r)\,\mathrm{e}^{-\mathrm{i}m\theta}\right).$$

Then, Eq. (2.21) can be reduced to

$$W = \int\limits_{\mathcal{D}_f}\sum_{m=0}^{\infty}\left(\frac{m^2}{r^2}H_0^2\left|\chi_m' - \frac{\chi_m}{r}\right|^2 + \left(m^2 - 1\right)\frac{m^2}{r^2}H_0^2|\chi_m|^2\right)\mathrm{d}x\,\mathrm{d}y. \tag{2.23}$$

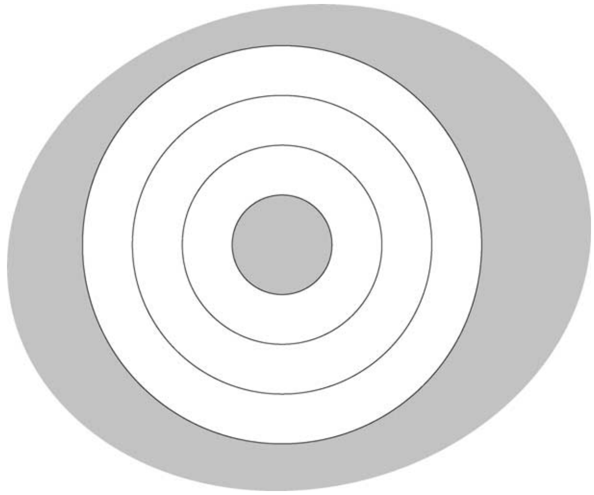

Fig. 2.

Here prime is used to denote the derivative with respect to r.

Evidently, W given by (2.23) is non-negative. We, therefore, may conclude that the circular cylinder in the equilibrium with circular $\mathbf{H}$-lines is always stable.

(B) A body in an potential magnetic field. Consider a perfectly conducting rigid body placed in a magnetostatic equilibrium with homogeneous magnetic field $\mathbf{H}_0 = H_0\mathbf{e}_x$ ($H_0 = \text{const}$). Since the body is perfectly conducting, the magnetic field cannot penetrate into it, so that the initial (homogeneous) magnetic field is distorted by the field produced by surface currents in the body to satisfy the boundary condition of no normal magnetic field at the body surface. The resulting magnetic field is potential, i.e., there are no currents in the fluid. We assume that the centre of mass of the body is fixed, but the body is free to rotate about any axis which passes through the fixed point.

The equilibrium magnetic field $\mathbf{H}(\mathbf{x})$ can be presented as

$$\mathbf{H} = \mathbf{H}_0 + \nabla\varphi_0, \tag{2.24}$$

where the potential $\varphi_0(\mathbf{x})$ is the unique solution of the following boundary-value problem:

$$\begin{aligned} &\nabla^2\varphi_0 = 0 \quad \text{in } \mathcal{D}_f, \\ &\mathbf{n}\cdot\nabla\varphi_0 = -\mathbf{n}\cdot\mathbf{H}_0 \quad \text{on } \partial\mathcal{D}_f, \\ &|\nabla\varphi_0| \to 0, \quad \text{as } |\mathbf{x}| \to \infty. \end{aligned} \tag{2.25}$$

(We assume that $\mathcal{D}_f$ is a simply connected domain.) The boundary value problem (2.25) is exactly the same as the hydrodynamic problem of an irrotational flow past a rigid body moving with the constant velocity $-\mathbf{H}_0$ provided that φ_0 is treated as the potential for the velocity field. It is well known (see, e.g., Batchelor [8]) that no force is exerted on the body by the fluid. It can be shown that this is also true for the analogous magnetostatic problem. Using this analogy, it can be shown that the total torque exerted on the body by the magnetic pressure is zero provided that one of the principal axes of body's virtual-mass tensor is parallel to the magnetic field at infinity $\mathbf{H}_0$ (a similar result for a body moving through an inviscid fluid can be found in Milne–Thomson [9, p. 529]). Recall that the virtual-mass tensor μ_{ik} is defined as (see, e.g., Batchelor [8])

$$\mu_{ik} = -\int\limits_{\partial\mathcal{D}_b} \varphi_i n_k \,\mathrm{d}S,$$

where the functions $\varphi_i(\mathbf{x})$ ($i = 1, 2, 3$) are the unique solutions of the following boundary-value problems:

$$\begin{aligned} &\nabla^2\varphi_i = 0 \quad \text{in } \mathcal{D}_f, \\ &\mathbf{n}\cdot\nabla\varphi_i = n_i \quad \text{on } \partial\mathcal{D}_b, \\ &|\nabla\varphi_i| \to 0, \quad \text{as } |\mathbf{x}| \to \infty. \end{aligned}$$

Evidently, the components of the tensor μ_{ik} depend only on the shape of the body and its orientation in space.

We study the stability of the equilibrium where one of the principal axes of body's virtual-mass tensor is parallel to $\mathbf{H}_0$. Without loss of generality we assume that the y and z axes are directed along other two principal axes (this always can be done simply by rotating our coordinate system about the x axis).

First, we decompose the perturbation magnetic field $\mathbf{h}$ into two parts:

$$\mathbf{h} = \mathbf{g} + \nabla\varphi. \tag{2.26}$$

Here $\nabla\varphi$ corresponds to the irrotational part of the perturbation magnetic field and is the unique solution of the following boundary-value problem:

$$\nabla^2\varphi = 0 \quad \text{in } \mathcal{D}_f, \qquad \mathbf{n}\cdot\nabla\varphi = \mathbf{n}\cdot[\boldsymbol{\xi}_b, \mathbf{H}] \quad \text{at } \partial\mathcal{D}_b, \qquad |\nabla\varphi| \to 0, \quad \text{as } |\mathbf{x}| \to \infty, \tag{2.27}$$

and $\mathbf{g}$ is an arbitrary divergence-free vector field satisfying the boundary conditions

$$\mathbf{g}\cdot\mathbf{n} = 0 \quad \text{at } \partial\mathcal{D} \text{ and } \partial\mathcal{D}_b.$$

This decomposition is the same as the decomposition that is usually used to prove Kelvin's minimum energy theorem in fluid mechanics (see, e.g., Batchelor [8]). It follows that

$$\int\limits_{\mathcal{D}_f} \mathbf{h}^2 \,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\mathcal{D}_f} \left\{(\nabla\varphi)^2 + \mathbf{g}^2\right\} \mathrm{d}x\,\mathrm{d}y. \tag{2.28}$$

Substitution of (2.28) into (2.18) yields

$$W = \frac{1}{2}\int\limits_{\mathcal{D}_f} \left((\nabla\varphi)^2 + \mathbf{g}^2\right) \mathrm{d}V + \frac{1}{2}\int\limits_{\partial\mathcal{D}_b} (\boldsymbol{\xi}_b\cdot\mathbf{n})(\boldsymbol{\xi}_b\cdot\nabla)\frac{\mathbf{H}^2}{2}\,\mathrm{d}S. \tag{2.29}$$

After standard but lengthy manipulations using the potential character of the basic magnetic field and Eqs. (2.27), formula (2.29) can be simplified to

$$W = \frac{1}{2}\int\limits_{\mathcal{D}_f} \mathbf{g}^2\,\mathrm{d}V + \frac{1}{2}(\mu_{22} - \mu_{11})H_0^2\phi_3^2 + \frac{1}{2}(\mu_{33} - \mu_{11})H_0^2\phi_2^2, \tag{2.30}$$

where μ_{11}, μ_{22} and μ_{33} are the virtual-mass coefficients corresponding to a motion of the body along the x-, y- and z-axes respectively.

It follows from (2.30) that *if in the equilibrium the principal axis of the virtual-mass tensor which corresponds to a minimum virtual mass is parallel to the magnetic field at infinity, then this equilibrium is stable; if this axis is perpendicular to the magnetic field at infinity, the equilibrium is unstable*. In particular, this result confirms the heuristic conclusion on the stability of an elliptic cylinder formulated in Introduction.

Note that we can guarantee the stability only with respect to infinitesimal rotations about the y- and z-axes, because ϕ_1, the angle of rotation about the x-axis, does not appear in Eq. (2.30). Rotations about the x-axis are less interesting in stability considerations because we have a family of equilibria which are obtained from the basic equilibrium simply by rotation of our coordinate system about the x-axis. Therefore, the variable ϕ_1 may be ignored.

2.4. Nonlinear stability of a body in a potential field

We now show that the stability criterion for a body in an irrotational magnetic field obtained in Subsection 2.3 guarantees not only linear but also genuine nonlinearly stability.

We consider finite-amplitude perturbations of the basic equilibrium. In the basic state, the body occupies the domain $\mathcal{D}_b$. In the perturbed state, the body occupies the domain $\widetilde{\mathcal{D}}_b(t)$ at time t. Let $\tilde{\mathbf{u}}$ be the velocity of the fluid, $\tilde{\mathbf{h}}$ the magnetic field, $\dot{\tilde{\mathbf{r}}}$ the velocity of the center of mass of the body and $\tilde{\boldsymbol{\sigma}}$ the angular velocity of the body in the perturbed state. The magnetic energy both in the perturbed and unperturbed states is infinite since $\tilde{\mathbf{h}} \to \mathbf{H}_0$ and $\mathbf{h} \to \mathbf{H}_0$ as $|\mathbf{x}| \to \infty$, where $\mathbf{H}_0 = H_0\mathbf{e}_x$ and $H_0 = \text{const}$. We may, however, consider the energy of the field $\tilde{\mathbf{h}} - \mathbf{H}_0$ produced solely by the surface currents in the body. The magnetic energy associated with $\tilde{\mathbf{h}} - \mathbf{H}_0$ is finite. It can be shown by a straightforward calculation, that the regularized energy $\widetilde{E}$, given by

$$\widetilde{E} = \widetilde{T} + \widetilde{W}, \quad \widetilde{T} = \frac{1}{2}\int\limits_{\widetilde{\mathcal{D}}_f(t)} \tilde{\mathbf{u}}^2\,\mathrm{d}V + \frac{1}{2}M\dot{\tilde{\mathbf{r}}}^2 + \frac{1}{2}I_{ik}\tilde{\sigma}_i\tilde{\sigma}_k, \quad \widetilde{W} = \frac{1}{2}\int\limits_{\widetilde{\mathcal{D}}_f(t)} \left(\tilde{\mathbf{h}} - \mathbf{H}_0\right)^2 \mathrm{d}V, \tag{2.31}$$

exists and is conserved by the exact (nonlinear) equations of motion provided that $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{h}} - \mathbf{H}_0$ decay at infinity faster than $|\mathbf{x}|^{-3/2}$. Therefore, we have

$$\widetilde{T}(t) + \widetilde{W}(t) = \widetilde{T}(0) + \widetilde{W}(0). \tag{2.32}$$

Since $\widetilde{T}$ is positive definite ($\widetilde{T} = 0$ if and only if $\tilde{\mathbf{u}} \equiv 0$, $\dot{\tilde{\mathbf{r}}} \equiv 0$ and $\tilde{\boldsymbol{\sigma}} \equiv 0$), we have

$$\widetilde{W}(t) \leqslant \widetilde{T}(0) + \widetilde{W}(0) = \widetilde{E}(0). \tag{2.33}$$

Note that inequality (2.33) is valid for any solution of the governing equations.

Consider now the magnetic energy $\widetilde{W}$. As in Subsection 2.3, we can decompose $\mathbf{h}' \equiv \tilde{\mathbf{h}} - \mathbf{H}_0$ into two parts

$$\mathbf{h}' = \nabla\tilde{\varphi} + \tilde{\mathbf{g}}, \tag{2.34}$$

where $\tilde{\varphi}$ is the unique solution of the boundary-value problem

$$\nabla^2\tilde{\varphi} = 0 \quad \text{in } \widetilde{\mathcal{D}}_f, \qquad \mathbf{n}\cdot\nabla\tilde{\varphi} = -\mathbf{n}\cdot\mathbf{H}_0 \text{ at } \partial\widetilde{\mathcal{D}}_b, \qquad |\nabla\tilde{\varphi}| \to 0, \quad \text{as } |\mathbf{x}| \to \infty, \tag{2.35}$$

and $\mathbf{g} = \mathbf{h}' - \nabla\tilde{\varphi}$. Note that $\nabla\cdot\mathbf{g} = 0$ in $\mathcal{D}_f(t)$ and $\mathbf{g}\cdot\mathbf{n} = 0$ at $\widetilde{\mathcal{D}}_b(t)$.

Again, the boundary value problem (2.35) is exactly the same as the hydrodynamic problem of an irrotational flow due to a rigid body moving through an inviscid fluid with constant velocity $-\mathbf{H}_0$. The solution of this problem is well-known (see, e.g., Batchelor [8]). Thus, for the magnetic energy we obtain the expression

$$\widetilde{W} = \frac{1}{2}\int\limits_{\widetilde{\mathcal{D}}_f(t)} \left(\tilde{\mathbf{g}}^2 + (\nabla\tilde{\varphi})^2\right) \mathrm{d}V = \frac{1}{2}\int\limits_{\widetilde{\mathcal{D}}_f(t)} \tilde{\mathbf{g}}^2\, \mathrm{d}V + \frac{1}{2}\tilde{\mu}_{ik}(t)H_{0i}H_{0k},$$

where $\tilde{\mu}_{ik}(t)$ is the virtual mass tensor corresponding to the orientation of the body in the perturbed state at time t. Hence,

$$\widetilde{W}(t) \geqslant \frac{1}{2}\tilde{\mu}_{ik}(t)H_{0i}H_{0k}. \tag{2.36}$$

If follows from (2.33) and (2.36) that

$$\frac{1}{2}\tilde{\mu}_{ik}(t)H_{0i}H_{0k} \leqslant \widetilde{T}(0) + \widetilde{W}(0) = \widetilde{E}(0). \tag{2.37}$$

Now we recall that in the basic state the magnetic field is potential and the (regularized) energy is given by

$$E = W = \frac{1}{2}\mu_{ik}H_{0i}H_{0k}.$$

Moreover, in the basic state μ_{ik} is diagonal ($[\mu_{ik}] = \operatorname{diag}\{\mu_{11}, \mu_{22}, \mu_{33}\}$), because we have assumed that the orientation of the body is such that one of the principal axes of the virtual-mass tensor is parallel to the magnetic field at infinity and the directions of the other two Cartesian axes is chosen so as to coinside with the other two the principal axes of the virtual-mass tensor.

Subtracting the (regularized) energy of the basic state from both sides of (2.37), we find that

$$\frac{1}{2}\left(\tilde{\mu}_{ik}(t) - \mu_{ik}\right)H_{0i}H_{0k} \leqslant \widetilde{E}(0) - E = \Delta E(0) \tag{2.38}$$

or, since $\mathbf{H}_0 = H_0\mathbf{e}_x$,

$$\frac{1}{2}\left(\tilde{\mu}_{11}(t) - \mu_{11}\right)H_0^2 \leqslant \Delta E(0). \tag{2.39}$$

The orientation of the body in the perturbed state can be viewed as the result of an initial rotation about the original z axis through an angle $\phi_3(t)$, a second rotation about the intermediate y axis through an angle $\phi_2(t)$, and a third rotation about the final x axis through an angle $\phi_1(t)$. Therefore, we can calculate the components of the tensor $\tilde{\mu}_{ik}(t)$ by applying the corresponding orthogonal transformations to μ_{ik} and obtain the formula

$$\begin{aligned} f(\phi_1,\phi_2,\phi_3) \equiv \tilde{\mu}_{11} - \mu_{11} = {}& (\mu_{22} - \mu_{11})(\sin\phi_1\sin\phi_2\cos\phi_3 - \cos\phi_1\sin\phi_3)^2 \\ & + (\mu_{33} - \mu_{11})(\cos\phi_1\sin\phi_2\cos\phi_3 + \sin\phi_1\sin\phi_3)^2. \end{aligned} \tag{2.40}$$

Hence,

$$f(\phi_1,\phi_2,\phi_3)H_0^2 \leqslant 2\Delta E(0). \tag{2.41}$$

We assume that

$$\mu_{22} > \mu_{11}, \qquad \mu_{33} > \mu_{11}, \tag{2.42}$$

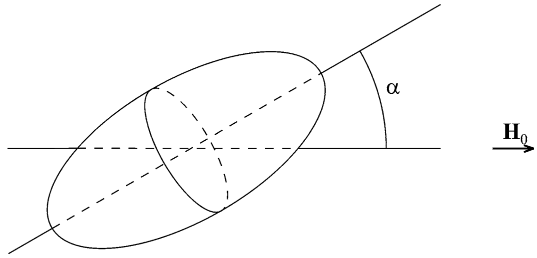

Fig. 3.

i.e., in the basic state the orientation of the body is such that the principal axis of the virtual-mass tensor that corresponds to the smallest virtual mass is parallel to the magnetic field at infinity. Then, function $f(\phi_1,\phi_2,\phi_3)$ is non-negative for all ϕ_1, ϕ_2 and ϕ_3. The point $\phi_1=\phi_2=\phi_3=0$ corresponds to the basic state. However, function $f(\phi_1,\phi_2,\phi_3)$ vanishes not only at this point, but also for all ϕ_1 if $\phi_2=\phi_3=0$. So, small values $f(\phi_1,\phi_2,\phi_3)$ may correspond to large values of ϕ_1, i.e., ϕ_1 cannot be controlled by $f(\phi_1,\phi_2,\phi_3)$. Function f does control the magnitude of ϕ_2 and ϕ_3, so that it is possible to prove the nonlinear stability with respect to variables ϕ_2 and ϕ_3.

Let

$$\gamma \equiv \min\{\mu_{22}-\mu_{11},\mu_{33}-\mu_{11}\}. \tag{2.43}$$

Our assumption (2.42) implies that γ is positive. From (2.40), we have

$$f \geqslant \gamma\big((\sin\phi_1\sin\phi_2\cos\phi_3-\cos\phi_1\sin\phi_3)^2+(\cos\phi_1\sin\phi_2\cos\phi_3+\sin\phi_1\sin\phi_3)^2\big)=\gamma\big(1-\cos^2\phi_2\cos^2\phi_3\big).$$

Let α be the angle between the axes associated with the virtual mass coefficient μ_{11} in the perturbed and unperturbed states (see Fig. 3). Then, $\cos\alpha=\cos\phi_2\cos\phi_3$, and we obtain

$$\gamma\big(1-\cos^2\alpha\big)\leqslant 2\Delta E(0). \tag{2.44}$$

Function $1-\cos^2\alpha$ in non-negative and has a strict minimum at $\alpha=0$, where it is zero. Therefore, for any $\epsilon>0$, there exists a $\delta>0$, such that

$$\Delta E(0)<\delta \quad\Rightarrow\quad \big|\alpha(t)\big|<\epsilon.$$

Thus, we have proved that *if inequalities* (2.42) *are satisfied, i.e., in the equilibrium the principal axis of the virtual-mass tensor that corresponds to a minimum virtual mass is parallel to the magnetic field at infinity, then this equilibrium is nonlinearly stable*.

Note that the stability in the above criterion is not the Lyapunov stability and should be understood in a wider sense. If it were the Lyapunov stability, we would use a single norm of the perturbation at $t=0$ and for $t>0$. Our condition uses two differet quantities to measure the amplitude of the perturbation at $t=0$ and for $t>0$: the initial perturbation is measured using the energy difference between the perturbed and unperturbed states and for $t>0$ the perturbation is measured by the angle associated with the orientation of the body relative to the direction of the magnetic field at infinity. This means that we can guarantee that the change in the orientation of the body at time t is small if not only the initial rotation of the body is small, but also the initial perturbation of the magnetic field is small.

3. Viscous fluids with infinite, finite or zero conductivity

In this section we show that the results obtained in Section 2 are also valid for a viscous fluid with finite electric conductivity. As before, we assume that the body is perfectly conducting. We start with the case of viscous, but still perfectly conducting fluid and then extend our analysis to fluids with finite conductivity.

3.1. Viscous, perfectly conducting fluid

The basic equilibrium of the system is the same as in Section 2 and is given by Eqs. (2.1)–(2.3). The linearised equations governing evolution of small perturbations to the basic state are (cf. Eqs. (2.14), (2.8), (2.9))

$$\boldsymbol{\xi}_{tt} = -\nabla p + \mathbf{J}\times\mathbf{h} + (\nabla\times\mathbf{h})\times\mathbf{H} + \nu\nabla^2\boldsymbol{\xi}_t, \quad \mathbf{h} = [\boldsymbol{\xi}, \mathbf{H}], \quad \nabla\cdot\boldsymbol{\xi} = 0, \tag{3.1}$$

$$M\ddot{r}_i = -\int_{\partial\mathcal{D}_b} \big(p + \mathbf{H}\cdot\mathbf{h} + \boldsymbol{\xi}_b\cdot\nabla\tilde{P}\big)n_i\,\mathrm{d}S + \nu\int_{\partial\mathcal{D}_b}(\mathbf{n}\cdot\nabla)\xi_{it}\,\mathrm{d}S, \tag{3.2}$$

$$I_{ik}\ddot{\phi}_k = -\int_{\partial\mathcal{D}_b} \big(p + \mathbf{H}\cdot\mathbf{h} + \boldsymbol{\xi}_b\cdot\nabla\tilde{P}\big)(\mathbf{x}\times\mathbf{n})_i\,\mathrm{d}S + \nu\int_{\partial\mathcal{D}_b} e_{ikl}x_k(\mathbf{n}\cdot\nabla)\xi_{lt}\,\mathrm{d}S, \tag{3.3}$$

where ν is the kinematic viscosity.

Boundary conditions (2.12) for $\boldsymbol{\xi}$ are replaced by the no-slip conditions

$$\boldsymbol{\xi} = 0 \quad \text{at } \partial\mathcal{D}, \qquad \boldsymbol{\xi} = \boldsymbol{\xi}_b \quad \text{at } \partial\mathcal{D}_b. \tag{3.4}$$

As in the previous section, boundary conditions for the magnetic field are automatically satisfied because of (3.4).

The quantity E, defined by Eq. (2.16), is not an invariant of the linearized problem (3.1)–(3.4). It can be shown, however, that E satisfies the equation

$$\dot{E} = -D, \quad D = \nu\int_{\mathcal{D}_f}\frac{\partial\xi_{it}}{\partial x_k}\frac{\partial\xi_{it}}{\partial x_k}\,\mathrm{d}V. \tag{3.5}$$

Integral D represents the rate of the energy dissipation due to viscosity and, evidently, is always nonnegative. Therefore, $E(t) \leqslant E(0)$, so that if E is positive definite for a given equilibrium, then this equilibrium is linearly stable. Thus, *the equilibria of the system that are linearly stable in the framework of inviscid model are also stable in the case of a viscous fluid*.

3.2. *Instability of a body in a viscous fluid*

We now show that positive definiteness of W is not only sufficient but also necessary for stability, i.e., an equilibrium is unstable if there are perurbations for which W, given by (2.17)/(2.18), is negative.

We employ the procedure of Vladimirov and Rumyantsev [5] (see also Barston [3]). First we introduce the notation

$$\begin{gathered} N = \frac{1}{2}\int_{\partial\mathcal{D}_f}\boldsymbol{\xi}^2\,\mathrm{d}V + \frac{1}{2}M\mathbf{r}^2 + \frac{1}{2}I_{ik}\phi_i\phi_i, \\ T = \frac{1}{2}\int_{\partial\mathcal{D}_f}\boldsymbol{\xi}_t^2\,\mathrm{d}V + \frac{1}{2}M\dot{\mathbf{r}}^2 + \frac{1}{2}I_{ik}\dot{\phi}_i\dot{\phi}_i, \qquad G = \nu\int_{\mathcal{D}_f}\frac{\partial\xi_i}{\partial x_k}\frac{\partial\xi_i}{\partial x_k}\,\mathrm{d}V. \end{gathered} \tag{3.6}$$

Assuming that $\boldsymbol{\xi}(\mathbf{x},t)$, $\mathbf{r}(t)$ and $\boldsymbol{\phi}(t)$ in Eqs. (3.6) represent a solution of the linearised equations (3.1)–(3.4), we diffirentiate N twice with respect to t. This yields the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{N} + G\big) = 2(T - W). \tag{3.7}$$

It follows from (3.5) and (3.7) that

$$\dot{E}_s = 2sE_s - 4sT_s - D_s, \tag{3.8}$$

where

$$\begin{aligned} E_s &= T_s + \frac{1}{2}sG + s^2N + W, \\ T_s &= \frac{1}{2}M(\dot{\mathbf{r}} - s\mathbf{r})^2 + \frac{1}{2}I_{ik}(\dot{\phi}_i - s\phi_i)(\dot{\phi}_k - s\phi_k) + \frac{1}{2}\int_{\mathcal{D}_f}(\boldsymbol{\xi}_t - s\boldsymbol{\xi})^2\,\mathrm{d}V, \\ D_s &= \nu\int_{\mathcal{D}_{f0}}\left(\frac{\partial u_i}{\partial x_k} - s\frac{\partial\xi_i}{\partial x_k}\right)\left(\frac{\partial u_i}{\partial x_k} - s\frac{\partial\xi_i}{\partial x_k}\right)\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3.9}$$

Let $s > 0$. Then, since T_s and D_s are always non-negative, it follows from (3.8) that

$$\dot{E}_s \leqslant 2sE_s.$$

Integrating this inequality over time, we obtain

$$E_s(t) \leqslant E_s(0)\,e^{2st}. \tag{3.10}$$

Note that (3.10) holds for any solution of the linearized problem (3.1)–(3.4) and for any positive s.

We assume that W can take negative values, i.e., there exists a set $\mathcal{Z}$ such that

$$W < 0 \quad \text{for } \{\boldsymbol{\xi}, \mathbf{r}, \boldsymbol{\phi}\} \in \mathcal{Z}, \qquad W \geqslant 0 \quad \text{for } \{\boldsymbol{\xi}, \mathbf{r}, \boldsymbol{\phi}\} \notin \mathcal{Z}. \tag{3.11}$$

We shall show that under this condition there exist solutions of the linearised problem that grow with time, and we shall obtain a lower bound for these solutions.

If the set $\mathcal{Z}$ is not empty, we can take initial values for $\boldsymbol{\xi}$, $\mathbf{r}$, $\boldsymbol{\phi}$ such that

$$\left\{\boldsymbol{\xi}(\mathbf{x},0), \mathbf{r}(0), \boldsymbol{\phi}(0)\right\} \in \mathcal{Z},$$

and therefore,

$$W(0) < 0. \tag{3.12}$$

Let us show that, in view of (3.12), it is always possible to choose $E_s(0) < 0$ (if it is so then exponential growth of perturbations follows directly from inequality (3.10)).

First we choose initial conditions $\boldsymbol{\xi}_t$, $\dot{\mathbf{r}}$ and $\dot{\phi}$ in such a way that

$$\boldsymbol{\xi}_t(\mathbf{x},0) = s\boldsymbol{\xi}(\mathbf{x},0), \qquad \dot{\mathbf{r}}(0) = s\mathbf{r}(0), \qquad \dot{\boldsymbol{\phi}}(0) = s\boldsymbol{\phi}(0). \tag{3.13}$$

With this choice, we have $T_s(0) = 0$ and therefore,

$$E_s(0) = \frac{1}{2}sG(0) + s^2N(0) + W(0).$$

It follows that $E_s(0) < 0$, provided that

$$0 < s < S, \quad S \equiv -\frac{G(0)}{4N(0)} + \sqrt{\left(\frac{G(0)}{4N(0)}\right)^2 - \frac{W(0)}{N(0)}}. \tag{3.14}$$

Obviously, $S > 0$ for any initial data which are consistent with condition $W(0) < 0$.

For any s from the interval, defined by (3.14), inequality (3.10) means that $E_s(t)$ is exponentially decreasing with time from its negative initial value $E_s(0)$, so that, in absolute value, $E_s(t)$ is growing. It can be shown that growth of $|E_s(t)|$ implies growth of $N(t)$. Thus, *an equilibrium is unstable if there are perturbations for which* $W(0) < 0$.

Evidently, the same result holds for an inviscid fluid. To see this, it suffices to pass to the limit $\nu \to 0$ in Eq. (3.14). The reason for this is that the mechanism of the instability has nothing to do with the viscosity and is simply the consequence of the existence of perturbations which reduce the energy of the system. If such an instability exists in a conservative system, usually it persists in a corresponding dissipative system, as it is the case here.

3.3. Viscous fluids with finite conductivity

We now consider a more realistic situation where the fluid has finite conductivity (but the body is still perfectly conducting). In this case, the equilibrium magnetic field $\mathbf{H}(\mathbf{x})$ satisfies Eqs. (2.1) and, in addition, the equation

$$\nabla \times \mathbf{J} = 0 \quad \text{in } \mathcal{D}_f. \tag{3.15}$$

We shall assume that in the basic equilibrium the magnetic field $\mathbf{H}$ is irrotational (i.e., $\mathbf{J} = 0$). Then, obviously, Eqs. (2.1) and (3.15) are both satified. In particular, the equilibria with potential magnetic field that were considered in Subsection 2.3 are possible for the fluid of finite conductivity.

The linearised equation are the same as Eqs. (3.1)–(3.3) except for the equation for the perturbation magnetic field which now has the form

$$\mathbf{h}_t = [\mathbf{u}, \mathbf{H}] - \frac{1}{\sigma}\nabla \times \mathbf{j}, \tag{3.16}$$

where σ is the electric conductivity of the fluid and $\mathbf{j} = \nabla \times \mathbf{h}$.

Boundary conditions for velocity and magnetic field are the same as before: no-slip condition for the velocity and no normal magnetic field at the rigid perfectly conducting boundaries $\partial\mathcal{D}_b$ and $\partial\mathcal{D}$. In addition to these conditions, for the fluid with finite conductivity we must prescribe two more boundary conditions at $\partial\mathcal{D}_b$ and $\partial\mathcal{D}$. These additional conditions are given by

$$\mathbf{n} \times (\mathbf{E} + \mathbf{v}_b \times \mathbf{H}) = 0 \quad \text{at } \partial\mathcal{D}_b, \qquad \mathbf{n} \times \mathbf{E} = 0 \quad \text{at } \partial\mathcal{D}, \tag{3.17}$$

where $\mathbf{E}$ is the perturbation electric field and, as before, $\mathbf{v}_b$ is the velocity of the body surface (note that, in view of no-slip condition for the velocity of the fluid, $\mathbf{v}_b = \mathbf{u}$). These conditions follow from the requirement that (in the reference frame, relative to which the surface is fixed) the tangent component of the electric field is zero. Combining (3.17) with Ohm's law for the fluid

$$\mathbf{j} = \frac{1}{\sigma}(\mathbf{E} + \mathbf{u} \times \mathbf{H}),$$

we find that

$$\mathbf{n} \times \mathbf{j} = 0 \quad \text{at } \partial\mathcal{D}_b \text{ and } \partial\mathcal{D}. \tag{3.18}$$

The 'energy' of the linearised equations has the form $E = T + W$ with T given by (3.6) and the 'potential energy' W given by

$$W = \frac{1}{2}\int_{\mathcal{D}_f} \mathbf{h}^2 \,\mathrm{d}V + \frac{1}{2}\int_{\partial\mathcal{D}_b} (\boldsymbol{\xi}_b \cdot \mathbf{n})(\boldsymbol{\xi}_b \cdot \nabla)\frac{\mathbf{H}^2}{2}\,\mathrm{d}S. \tag{3.19}$$

Note that in the case of finite conductivity the relation $\mathbf{h} = [\boldsymbol{\xi}, \mathbf{H}]$ is not valid. Formal integration of Eq. (3.16) over t yields

$$\mathbf{h} = [\boldsymbol{\xi}, \mathbf{H}] - \frac{1}{\sigma}\nabla \times \mathbf{q}, \tag{3.20}$$

where

$$\mathbf{q}(\mathbf{x}, t) = \int_0^t \mathbf{j}(\mathbf{x}, t')\,\mathrm{d}t' + \mathbf{q}_0(\mathbf{x}). \tag{3.21}$$

Differentiating E with respect to time, we obtain

$$\dot{E} = -D, \quad D = \nu \int_{\mathcal{D}_f} \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}\,\mathrm{d}V + \frac{1}{\sigma}\int_{\mathcal{D}_f} \mathbf{j}^2\,\mathrm{d}V. \tag{3.22}$$

This equation is the same as (3.5) except that now D contains an additional term due to resistive dissipation of energy. Since D is always nonnegative, we have $E(t) \leqslant E(0)$, so that if W is positive definite for a given equilibrium, then so is E, and this equilibrium is linearly stable. Thus, *the equilibria of a body in a potential magnetic field that are stable for inviscid, perfectly conducting fluid are also stable in the case of a viscous fluid with finite conductivity*.

It can be shown that, in the case of finite conductivity, Eq. (3.7) is still valid provided that G is given by (cf. Eq. (3.6))

$$G = \nu \int_{\mathcal{D}_f} \frac{\partial \xi_i}{\partial x_k}\frac{\partial \xi_i}{\partial x_k}\,\mathrm{d}V + \frac{1}{\sigma}\int_{\mathcal{D}_f} \mathbf{q}^2\,\mathrm{d}V. \tag{3.23}$$

Now we can repeat the arguments of Subsection 3.2 and arrive at the same conclusion that an equilibrium is unstable provided the 'potential energy' can take negative values.

Thus, we have shown that necessary and sufficient conditions for linear stability obtained in Section 2 for a body in an potential magnetic field are also valid in the case of a viscous fluid with finite conductivity.

3.4. Body in a non-conducting viscous fluid

Consider now the situation where the fluid has zero conductivity. Now there are no electric currents in the fluid and the magnetic field is produced only by external currents and surface currents in the body, so that the magnetic field in the flow domain $\mathcal{D}_f$ is potential. We consider the stability of the same equilibria of a perfectly conducting body in a potential magnetic field as in Subsection 3.3. The linearised equations and boundary conditions for the fluid reduce to

$$\begin{aligned} &\boldsymbol{\xi}_{tt} = -\nabla p + \nu\nabla^2\boldsymbol{\xi}_t, \quad \nabla \cdot \boldsymbol{\xi} = 0 \quad \text{in } \mathcal{D}_f, \\ &\boldsymbol{\xi} = 0 \quad \text{at } \mathcal{D}, \qquad \boldsymbol{\xi} = \boldsymbol{\xi}_b \quad \text{at } \mathcal{D}_b. \end{aligned} \tag{3.24}$$

Equation for the perturbation magnetic field $\mathbf{h} = \nabla\varphi$ simplifies to the Laplace equation

$$\nabla^2\varphi = 0 \quad \text{in } \mathcal{D}_f, \tag{3.25}$$

supplemented with the boundary conditions

$$\mathbf{n} \cdot \nabla\varphi = 0 \quad \text{at } \partial\mathcal{D}, \qquad \mathbf{n} \cdot \nabla\varphi = \mathbf{n} \cdot [\boldsymbol{\xi}_b, \mathbf{H}] \quad \text{at } \partial\mathcal{D}_b. \tag{3.26}$$

The equations governing the displacement and rotation of the body are given by (3.2) and (3.3).

The 'energy' of the linearised equations is $E = T + W$ where T given by (3.6) and

$$W = \frac{1}{2}\int_{\mathcal{D}_f} (\nabla\varphi)^2 \,\mathrm{d}V + \frac{1}{2}\int_{\partial\mathcal{D}_b} (\boldsymbol{\xi}_b\cdot\mathbf{n})(\boldsymbol{\xi}_b\cdot\nabla)\frac{\mathbf{H}^2}{2}\,\mathrm{d}S. \tag{3.27}$$

It can be shown that Eqs. (3.5) and (3.7) are valid and the same analysis as in Subsections 3.1 and 3.2 lead us to the conclusion that an equilibrium of stable if W is positive definite and unstable if there exist perturbations for which W is negative. Hence, the stability criterion for a body in a potential field obtained in Subsection 2.3 is also applicable to the case of non-conducting fluid.

4. Conclusion

We have shown that the energy principle of Bernstein et al for magnetostatic equilibria can be extended to include a rigid perfectly conducting body suspended in a viscous fluid which may be perfectly conducting, have finite conductivity or be non-conducting. The energy principle provides both necessary and sufficient conditions for the linear stability. For a body in an irrotational magnetic field, we have proved that the linear stability criterion gives, in fact, the condition for genuine nonlinear stability.

Physical mechanism of stability and instability is very transparent: if there are perturbations which reduce the energy, then we have instability, if there are no such perturbations, the considered basic state is stable. If instability occurs, it is the potential energy relesed by the perturbation that is driving the instability. As was discussed in Introduction, the question about instability is more complicated: even in finite-dimensional systems, there are situations when the existence of perturbations which reduce the energy does not imply instability. For equilibrua of a body immersed in a fliud we have proved that the energy principle holds and an equilibrium is unstable in the case of no energy minimum at this equilibrium. We have also shown that the dissipation effects cannot arrest the instability (though they can slow down its development). The inclusion of these effects makes the problem that at first looks somewhat artificial closer to real life phenomena. Let us discuss the domain of applicability of the results in more detail.

Throughout the paper, we assumed that the body is perfectly conducting. This means that our theory is applicable to real life phenomena only if the electric conductivity of the body is much larger than that of the fluid. This restriction is always satisfied at least for a non-conducting fluid. For conducting fluids with finite electric conductivity, our results hold for times of order of the diffision time for the magnetic field in the body.

If we want to observe the instability of the body in an experiment, we must also satisfy the condition that the characteristic time of the diffusion of the magnetic field in the body is much larger than the characteristic time of the development of the instability. The latter, in turn, depends on the viscosity of the fluid. Although viscosity does not prevent instability it slows down the growth rate of perturbations. Therefore, the viscosity should be relatively small. Thus, an experimental realization of the instability predicted in the paper may be possible for a body with sufficiently high electric conductivity suspended in a conducting (or non-conducting) fluid whose viscosity is relatively small.

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